MTH 605: Topology I

Practice Assignment I

1. If $A \subset X$, a *retraction* of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

2. Define an equivalence relation $\sim$ on $\mathbb{R}^2$ as follows: $(x_0, y_0) \sim (x_0, y_0)$ if $x_0 + {y_0}^2 = x_1 + {y_1}^2$. Describe the corresponding quotient space X^*.

3. A *topological group* is a group $(G, \cdot)$ that is also a topological space satisying the T_1 axiom, such that the group operation $(g, h) \mapsto g \cdot h$ and the map $g \mapsto g^{-1}$ are both continuous maps. Show that $(\mathbb{R}, +)$, $\mathrm{GL}(n)$, and S^1 (seen as a subset of $\mathbb{C}$) are topological groups.

4. Let G be a topological group, and let H be a subspace and a subgroup of G.
   (a) Show that both H and $\bar{H}$ are topological groups.
   (b) Give G/H the quotient topology using the lest cosets as partitions. Show that if H is closed in G, then the singletons are closed in G/H.
   (c) Show that $G \to G/H$ is open.
   (d) Show that if H is closed and $H \trianglelefteq G$, then G/H is a topological group.
   (e) Using (d), show that $\mathbb{R}/\mathbb{Z}$ is a topological group. Describe this space.

5. If τ and τ' be two topologies on X such that $\tau \subset \tau'$. What does the connected of X in one topology imply in the other?

6. A space is *totally disconnected* if its only connected subsets are the one-point sets. Show that if X has the discrete topology, then X is totally disconnected.

7. Determine whether the following spaces are connected.
   (a) An infinite set with the cofinite topology.
   (b) $\mathbb{R}_\ell$.

8. Let $p : X \to Y$ is a quotient map each of whose fibers are connected. Show that X is connected, whenever Y is connected.

9. Using connectedness, establish the following facts.
   (a) $(0, 1)$, $(0, 1]$, and $[0, 1]$ are not homeomorphic.
   (b) $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic for $n > 1$.

10. Show that if $f : [0, 1] \to [0, 1]$ is a continuous map, then f has a fixed point.